

## Field theory and second renormalization group for multifractals in percolation

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The field theory for multifractals in percolation is reformulated in such a way that multifractal exponents clearly appear as eigenvalues of a second renormalization group. The first renormalization group describes geometrical properties of percolation clusters, while the second one describes electrical properties, including noise cumulants. In this context, multifractal exponents are associated with symmetry-breaking fields in replica space. This provides an explanation for their observability. It is suggested that multifractal exponents are “dominant” instead of “relevant” since there exists an arbitrary scale factor which can change their sign from positive to negative without changing the physics of the problem.

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### I. INTRODUCTION

The renormalization group and critical phenomena have provided, over the years, key concepts that allow us to understand problems ranging from phase transitions to percolation. Even though infinite sets of exponents, such as crossover exponents, were calculated early after the introduction of the renormalization group in critical phenomena [1], attention is usually focused on a few relevant exponents. This focus on a few exponents is justified since observable quantities in general couple to many renormalization group eigenoperators, including the most relevant ones, which eventually dominate their behavior. That lore was challenged relatively recently by the appearance of infinite sets of measurable exponents in various fields. Problems where such infinite sets occur are collectively known as multifractal problems [2–4], even though some of these problems have a quite different physical nature. We discuss here the problem of electrical properties of percolating networks [5–7].

The difficulties associated with the formulation of a Lagrangian field theory for multifractals were discussed by Ludwig and Duplantier [8,9]. In particular, the exponents  $x_n$  describing multifractal behavior are a convex function of  $n$ , while the analogous exponents of field powers in a field theory are in general concave as a consequence of stability and correlation inequalities [8]. But, as stated in Ref. [8], powers of composite operators, such as derivatives, can exhibit multifractal behavior. It is therefore possible to formulate for multifractals a field theory that is amenable to a renormalization group analysis. This has been done for percolation by Parks, Harris, and Lubensky [10] (PHL). This field theory, however, does have some peculiarities that make it different from usual field theories. In the present paper, we reinterpret the field theory of PHL so as to make special features

clearer. In fact, there are also some differences in the way we set up and interpret the field theory. A detailed discussion of the differences and of the reasons that motivate our approach have appeared [11]. The structure that will emerge here is that geometrical properties of percolation clusters are described by a standard field theory on which one can perform a renormalization group analysis. We call this the first renormalization group. By contrast, multifractal properties, originating from the electrical properties of the cluster, are described by a second renormalization group [12,13]. The structure of this second renormalization group depends on the first renormalization group: Once the usual recursion relations for percolation are derived, the recursion relation for each multifractal moment must be found by a further projection onto an appropriate eigenbasis. In the second renormalization group, the role of the fields is played by the microscopic noise cumulants  $v_s$ . These fields  $v_s$  are conjugate to replica-space gradients of the operators  $\Phi$ , for example,  $\mathbf{k}^{2s} \Phi_{\mathbf{k}}(\mathbf{q}) \Phi_{-\mathbf{k}}(-\mathbf{q})$ , where  $\mathbf{q}$  is defined in the usual Brillouin zone while  $\mathbf{k}$  is the replica-space extension of the Fourier variable for the voltage. Without changing the physics, the scale of  $\mathbf{k}$  can be changed by an arbitrary factor. The existence of this arbitrary scale factor is a direct consequence of the linearity of Kirchhoff’s laws. This has no equivalent in usual critical phenomena. Due to this arbitrary scale, the multifractal exponents can change sign, rendering inappropriate the usual classification as relevant or irrelevant. We suggest, therefore, calling the multifractal exponents dominant exponents since they determine the leading scaling behavior of observable quantities while corrections would come from subdominant exponents. The fields for multifractal moments are associated with operators that break rotational (permutation [8]) symmetry in replica space. This seems to be the reason why the multifractal moments are each asso-

ciated with a different dominant exponent and not just to a few exponents.

In the following section, we provide a short review of the phenomenology of multifractals in percolation. Then, in Sec. III, we derive the field theory and, in Sec. IV, proceed with the renormalization groups. The discussion in Sec. V clarifies some points of interpretation.

## II. PHENOMENOLOGY OF MULTIFRACTALS IN PERCOLATION

For completeness, let us recall how the infinite set of exponents appears in percolation through the problem of noise. Suppose that the conducting resistors of a percolating network are fluctuating independently in time. The total resistance of a given network is then a random variable in time whose cumulants depend on those of each component resistor. The cumulants of a given order are assumed to be the same for all component resistors. The cumulants of the total resistance  $R$  are, in principle, accessible experimentally and measurements of the second cumulant, corresponding to  $1/f$  noise, have actually been performed [14]. Theoretically, for systems of finite size  $L$  at bulk criticality, one finds, after averaging over the microscopic noise, that the cumulants  $C_R^{(n)}(L)$  of order  $n$  scale as

$$\langle C_R^{(n)}(L) \rangle_C = v_n \left\langle \sum_{\alpha} i_{\alpha}^{2n} \right\rangle_C \sim L^{-x_n}, \quad (1)$$

where  $C$  represents average over percolating lattice configurations,  $i_{\alpha}$  is the current that flows in branch  $\alpha$  of the time averaged network, and  $v_n$  is the amplitude of the  $n$ th cumulant of the elementary resistance fluctuations. For example, the usual electrical noise amplitude for one microscopic resistance  $r$  is obtained from

$$v_2 = \{ \delta r \delta r \}_f, \quad (2)$$

where  $\{ \}_f$  is an ensemble average that is equivalent to the time average over the noise. The first equality in Eq. (1) follows from the assumption that different elementary resistors are uncorrelated and from Cohn's theorem

$$\delta R = \sum_{\alpha} \delta r_{\alpha} i_{\alpha}^2, \quad (3)$$

where the total input current  $I_{\text{inj}}$  is unity. Each exponent  $x_n$  in Eq. (1) is different and is not a simple linear function of  $n$ , as commonly occurs in critical phenomena under the name gap scaling [1]. Such an infinite set of exponents also arises in the other analogous problems mentioned above. In all these problems, one is assigning to parts of a fractal network a weight, or a measure, that is obtained from the solution of Laplace's equation. Here that weight corresponds to the power dissipated in a bond.

Finally, let us recall that the positive integer moments  $\langle \sum_{\alpha} i_{\alpha}^{2n} \rangle_C$  suffice to characterize completely the distribution of the currents flowing through the network [15]. But, as stated previously, the *exact* value of the integer moments is necessary to reconstruct all the information so that the leading scaling behavior of the positive

moments does not suffice to find, for example, negative moments. In the following, we concentrate only on the scaling behavior of the positive integer moments. Note that in order to keep the same notation as PHL [10], we work with exponents defined by

$$\psi_s/\nu \equiv -x_n \quad (s = n). \quad (4)$$

## III. FIELD THEORY FOR THE GENERATING FUNCTION

### A. Generating function for the cumulants

Since Kirchhoff's law can be obtained by minimizing the entropy production, or the dissipated power in the electrical network, it is natural to start from a generating function for the resistance between two points  $y$  and  $y'$  of the network

$$W(k; y, y') = \int_{\Delta V_{\min}}^{\Delta V_{\max}} d[V(x)] \times \exp \{ -H + ik [V(y) - V(y')] \}, \quad (5)$$

where  $\Delta V_{\min}$  and  $\Delta V_{\max}$  are, respectively, the minimum and the maximum voltage drops for a finite size system and  $H$  is given by

$$H \equiv \frac{1}{2} \sum_{(x, x')} \sigma_b(x, x'; C, f) [V(x) - V(x')]^2 \quad (6)$$

with  $\sigma_b(x, x'; C, f)$  the conductivity of the bonds  $\alpha$  linking each pair of points  $x$  and  $x'$  for a given configuration  $C$  of the random resistor network. For each given configuration of the random network, the conductivities fluctuate in time. In other words,

$$\sigma_b(x, x'; C, f) = \sigma_0(x, x'; C)(1 + \epsilon), \quad (7)$$

where  $\epsilon \ll 1$  is a random variable whose probability distribution is given by  $f(\epsilon)$ . Hence there are two types of averages to perform: the usual bond-disorder average and, for each lattice configuration, an average over the microscopic noise. To obtain a Hamiltonian with the same structure as for a spin system [e.g.,  $J(x - x')S(x)S(x')$ ] one uses Fourier transforms. Since the potential differences are on a bounded interval, determined by the boundary conditions, it is possible to use discrete Fourier series. Formally, we may write

$$H = \sum_{(x, x')} \sigma_b(x, x'; C, f) \sum_k A_k e^{ik[V(x) - V(x')]}, \quad (8)$$

where the Fourier coefficients are given by

$$A_k = \frac{1}{\Delta V_{\max}} \int_{\Delta V_{\min}}^{\Delta V_{\max}} d(\Delta V) [\Delta V]^2 e^{ik\Delta V}. \quad (9)$$

Because the  $k = 0$  terms correspond to a uniform distribution of voltage, they will be discarded. In the theory of Stephen [16], the order parameter  $\psi$  is defined in

such a way that its autocorrelation function vanishes in the nonpercolating phase and decays exponentially in the percolating phase. This order parameter is

$$\psi_{\mathbf{k}}(y) \equiv e^{ikV(y)}, \quad (10)$$

where the wave vector  $k$  takes a discrete number of values  $2\pi n/(\Delta V_{\max})$ . The presence of a lattice instead of a continuum leads to the existence of an ultraviolet cutoff  $\Lambda_k$  corresponding to the minimum voltage [17]  $\Lambda_k = 2\pi/(\Delta V_{\min})$ . That cutoff becomes infinite in the percolation limit  $\sigma_0^{-1} \rightarrow 0$ . For a given realization, we may then write  $H$  in the form mentioned, namely,

$$H = \sum_{\mathbf{k}} A_{\mathbf{k}} \sum_{\langle x, x' \rangle} \sigma_b(x, x'; C, f) \psi_{\mathbf{k}}(x) \psi_{\mathbf{k}}(x'), \quad (11)$$

where, in the second sum, we consider only the connected links of the network. To understand what follows it is important to realize that the  $k$ 's scale as  $\Delta V_{\max}^{-1}$  and that we must use a discrete Fourier expansion since the replica method is only valid for a finite system. The infinite-system limit is taken at the very end.

In the limit where  $\sigma_0^{-1} \rightarrow 0$ , the saddle point configuration for the voltage drops obeys Kirchoff's laws and, in this limit, we obtain for the generating function

$$W(k; y, y') = \langle \psi_{\mathbf{k}}(y) \psi_{-\mathbf{k}}(y') \rangle = Z e^{-\frac{1}{2} k^2 R(y, y'; C, f)}, \quad (12)$$

where

$$\begin{aligned} Z &= \int d[V(x)] \\ &\times \exp\left(-\frac{1}{2} \sum_{\langle x, x' \rangle} \sigma_b(x, x'; C, f) [V(x) - V(x')]^2\right) \\ &= \int d[V(x)] \exp(-H) \end{aligned} \quad (13)$$

and  $R(y, y'; C, f)$  is the resistance between nodes  $y$  and  $y'$  for a given configuration  $C$  of the random network and realization  $f$  of the noise. As in the work of PHL [10], the cumulant averages for the resistance noise may be obtained from the generating function. We first average Eq. (13) over the probability distribution for the noise  $f$  and then we average over the disorder as follows:

$$\left\langle \left\{ \exp\left(-\frac{1}{2} k^2 R(y, y'; C)\right) \right\}_f \nu(y, y'; C) \right\rangle_C, \quad (14)$$

where the function  $\nu$  plays the role of a conditional probability that the two points  $y$  and  $y'$  are connected. Expanding the left-hand side in cumulants of the resistance, Eq. (14) becomes

$$\begin{aligned} &\left\langle \exp\left[\sum_{s \geq 1} \frac{(-1)^s}{2^s s!} k^{2s} C_R^{(s)}(y, y'; C)\right] \nu(y, y'; C) \right\rangle_C \\ &= \left\langle \{ \langle \psi_{\mathbf{k}}(y) \psi_{-\mathbf{k}}(y') \rangle \}_f \right\rangle_C. \end{aligned} \quad (15)$$

The cumulants of the resistance fluctuations  $C_R^{(s)}(y, y'; C)$  may thus be obtained from derivatives of the generating function since, by Eq. (1) and in the  $\sigma_0^{-1} \rightarrow 0$  limit, they are proportional to the microscopic cumulants [5]

$$\begin{aligned} &\left\langle C_R^{(s)}(y, y'; C) \nu(y, y'; C) \right\rangle_C \\ &= (-1)^s 2^s s! k^{-2s} v_s \\ &\times \frac{\partial \left[ \left\langle \{ \langle \psi_{\mathbf{k}}(y) \psi_{-\mathbf{k}}(y') \rangle \}_f \right\rangle_C \right]}{\partial v_s} \Big|_{v_l=0 \forall l}. \end{aligned} \quad (16)$$

Thus the macroscopic cumulants of the noisy resistor network can be obtained from the autocorrelation of the order parameter, after averaging over bond disorder. It is important to note that the quantities  $v_s$  play the role of fields whose conjugate operators will contain polynomials in  $k$ , as explained later.

## B. Replica method and effective action

Since there are two types of averages over random variables, we introduce two types of replicas, as suggested by PHL [10]:  $N$  replicas for the average over noise ( $f$ ) and  $M$  replicas for the average over bond disorder ( $C$ ):

$$\left\langle \{ \ln Z \}_f \right\rangle_C = \lim_{N, M \rightarrow 0} \frac{\left( \left\langle \left[ \{ Z^N \}_f \right]^M \right\rangle_C - 1 \right)}{NM}. \quad (17)$$

As usual, the limiting process is justified only for finite systems. In other words, the limit  $L \rightarrow \infty$  is taken after the limits  $N, M \rightarrow 0$ .

We do not repeat the details of the derivation of the field theory [10]. As usual it proceeds by introducing Hubbard-Stratonovich variables  $\Phi$  conjugate to each  $\psi$  appearing in  $Z^{NM}$ . Expanding in powers of  $\psi$  and performing the integrals over the voltages appearing in the original generating function, one generates a power series in  $\Phi$  which can be reexponentiated to yield an effective action for the  $\Phi$ . These variables are now the *operators* of the field theory. For Hubbard-Stratonovich transformations, the generating function  $\langle \psi_{\mathbf{k}}(x) \psi_{-\mathbf{k}}(x) \rangle$  is simply proportional to  $\langle \Phi_{\mathbf{k}}(\mathbf{x}) \Phi_{-\mathbf{k}}(\mathbf{x}) \rangle$ ; hence all we need is the effective action for the operators  $\Phi_{\mathbf{k}}$

$$\begin{aligned} \mathcal{L}(\Phi) &= \frac{1}{2} \int d^d x \Delta k \sum_{\mathbf{k}} r_{\mathbf{k}} [\Phi_{\mathbf{k}}(\mathbf{x}) \Phi_{-\mathbf{k}}(\mathbf{x}) \\ &+ \nabla \Phi_{\mathbf{k}}(\mathbf{x}) \cdot \nabla \Phi_{-\mathbf{k}}(\mathbf{x})] \\ &+ \frac{u_3}{6} \int d^d x (\Delta k)^2 \sum_{\mathbf{k}_1 + \mathbf{k}_2 \neq 0} \Phi_{\mathbf{k}_1}(\mathbf{x}) \\ &\times \Phi_{\mathbf{k}_2}(\mathbf{x}) \Phi_{-\mathbf{k}_1 - \mathbf{k}_2}(\mathbf{x}). \end{aligned} \quad (18)$$

The  $\mathbf{k}$  in Eq.(18) is any of the  $\mathbf{k}$  Fourier variables whose components are labeled  $k_{\alpha\beta}$  in the  $NM$ -dimensional replica space and  $\mathbf{k}^2 \equiv \sum_{\alpha\beta} k_{\alpha\beta}^2$  is the square modulus of  $\mathbf{k}$ . There are  $(L^{MN} - 1)$  operators  $\Phi_{\mathbf{k}}(x)$  at a spatial

point  $x$  (the  $\mathbf{k} = \mathbf{0}$  case is omitted). The  $k_{\alpha\beta}$  are conjugate to the electrical potentials of the replicated systems. They contain a scale factor corresponding to the scale of the electrical potential. In the limit of geometrical percolation (i.e., no transport property), the  $k_{\alpha\beta}$  are all zero. Note that despite the notation, the  $\mathbf{k}$  are tensors of rank one and not two, as far as rotations in replica space are concerned. As usual, to find critical exponents the  $\mathbf{k}$  dependence of the coupling constant  $u_3$  can be neglected, but that of  $r_{\mathbf{k}}$  is crucial. The scaling of the terms entering  $r_{\mathbf{k}}$  can be inferred from the scaling properties of the generating function. Writing explicitly the dependence on the microscopic cumulants  $v_s$ , one obtains for the scaling properties of the generating function,

$$G_{\mathbf{k}}(y - y', p - p_c, \{v_s\}_{s \geq 1}, I_{\text{inj}}) = G_{\lambda \mathbf{k}}(y - y', p - p_c, \{\lambda^{-2s} v_s\}_{s \geq 1}, I_{\text{inj}}) \quad (19)$$

since  $\mathbf{k}$  scales as  $\frac{1}{\Delta V_{\text{max}}}$ . Indeed, the cumulants  $C_R^{(s)}(y, y'; C)$  are linearly proportional to the microscopic cumulants  $v_s$  so that, as can be seen from Eq.(15), the autocorrelation function for the order parameter does have the scaling property (19). Expanding for small  $v_s$ , the most general form for the  $r_{\mathbf{k}}$  is then a homogeneous polynomial in  $\mathbf{k}$  and  $v_s$ ,

$$r_{\mathbf{k}} = p - p_c + \sum_{s \geq 1} \mathbf{k}^{2s} \left( \sum_{s_1 + s_2 + \dots + s_i = s} v_{s_1} v_{s_2} \dots v_{s_i} \times P_{(s_1, s_2, \dots, s_i)}(\dots, \theta_{\mathbf{k}_{\alpha, \beta}}, \dots) \right). \quad (20)$$

In Eq. (20) the functions  $P_s(\dots, \theta_{\mathbf{k}}, \dots)$  depend on the angular variables  $\theta_{\mathbf{k}}$  in the  $(n = MN)$ -dimensional replica space. For each  $\mathbf{s} = (s_1, s_2, \dots, s_i)$ , these polynomials can be expressed as a linear combination of the spherical harmonics for the  $MN$  angular variables  $\theta_{\mathbf{k}}$ . Their explicit expressions depend on the particular distribution  $f$  for the noise of the elementary bond resistor. The expansion in Eq. (20) is justified by the fact that for a finite system, the effective action is an analytic function of  $\mathbf{k}$  and hence, by scaling arguments, of  $v_s$ . For  $s = 1$ , it will be seen in the Appendix that  $P_1(\theta_{\mathbf{k}}) = 1$ , i.e.,  $\mathbf{k}^2$  is the only polynomial of degree 2 that contributes and it is an eigenpolynomial of the second renormalization group.

Finally, observable quantities are obtained using the standard replica-method identity

$$G_{\mathbf{k}}(y - y', p - p_c, \{v_s\}_{s \geq 1}, I_{\text{inj}}) = \left\langle \{ \langle \psi_{\mathbf{k}}(y) \psi_{-\mathbf{k}}(y') \rangle \}_f \right\rangle_C \sim \lim_{N, M \rightarrow 0} \langle \Phi_{\mathbf{k}}(y) \Phi_{-\mathbf{k}}(y') \rangle. \quad (21)$$

In the absence of symmetry breaking in replica space, the modulus of the replicated  $\mathbf{k}$  in Eq. (21) is equal to  $k$  in expression (15).

### C. Scaling properties

Near the percolation critical point, the generating function scales as

$$\lim_{L \rightarrow \infty; \sigma_0^{-1}, n \rightarrow 0} G_{\mathbf{k}}(y - y', p - p_c, \{v_s\}_{s \geq 1}, I_{\text{inj}}) = \lim_{L \rightarrow \infty; \sigma_0^{-1}, n \rightarrow 0} \lambda^{2 - \eta_p - d} \times G_{\mathbf{k}}((y - y') \lambda^{-1}, (p - p_c) \lambda^{1/\nu_p}, \{v_s \lambda^{\psi_s/\nu_p}\}_{s \geq 1}, I_{\text{inj}}), \quad (22)$$

which also implies

$$\lim_{L \rightarrow \infty; \sigma_0^{-1}, n \rightarrow 0} G_{\mathbf{k}}(y - y', p - p_c, \{v_s\}_{s \geq 1}, I_{\text{inj}}) = (p - p_c)^{(d - 2 + \eta_p)\nu_p} S((y - y')(p - p_c)^{\nu_p}, \{v_s (p - p_c)^{-\psi_s}\}_{s \geq 1}), \quad (23)$$

where  $S$  is a scaling function. As usual, the correlation length  $\xi$  behaves as  $(p - p_c)^{-\nu}$ . The  $L \rightarrow \infty$  limit must be taken at the end for the replica method to be valid. From the way the problem is set up, the  $n \rightarrow 0$  limit must be taken before the  $\sigma_0^{-1} \rightarrow 0$  limit. However, as we shall see in the following sections, we need to add another condition to linearize the second renormalization group equations, namely, that

$$v_s \ll v_1 \quad \forall s > 1.$$

This inequality is consistent with the phenomenology of  $1/f$  noise.

## IV. RENORMALIZATION GROUP APPROACH

### A. First and second renormalization group

The renormalization group equations are obtained [10] from the usual procedure for a cubic Landau-Ginzburg

functional. Using the Wilson approach, operators  $\Phi_{\mathbf{k}}(\mathbf{q})$  whose wave vector  $\mathbf{q}$  is in a shell  $\Lambda/b < q \leq \Lambda$ , near the cutoff  $\Lambda$  coming from the physical lattice, are traced over. It is always possible to choose the lattice spacing units such that  $\Lambda = 1$ . Lengths are then rescaled by  $b$ , while the operators are rescaled as

$$\Phi'_{\mathbf{k}}(\mathbf{q}' = b\mathbf{q}) = b^{(-d-2+\eta_p)/2} \Phi_{\mathbf{k}}(\mathbf{q}). \quad (24)$$

Let  $K_d$  be the surface of the  $d$ -dimensional sphere. To one-loop order, in dimension  $d = 6 - \epsilon$  near the upper-critical dimension  $d = 6$ , the differential recursion relations for  $r_{\mathbf{k}}$  and for the coupling constant  $g = K_d u_3^2/2$  take the form [10,18]

$$\frac{dr_{\mathbf{k}}}{dl} = (2 - \eta_p) r_{\mathbf{k}} - g \Sigma_{\mathbf{k}}, \quad \frac{dg}{dl} = (\epsilon - 3\eta_p) g - 8g^2, \quad (25)$$

where  $l$  is defined by  $b = e^l$ . In the first equation,  $\Sigma_{\mathbf{k}}$  is the self-energy-correction to one loop [10]

$$\begin{aligned} \Sigma_{\mathbf{k}} &= \lim_{n \rightarrow 0} (\Delta k) \sum_{\mathbf{p}, \mathbf{p}+\mathbf{k} \neq 0} G[\mathbf{p}] G[\mathbf{p} + \mathbf{k}] \\ &\equiv -2G[\mathbf{k}] G[0] + \tilde{\Sigma}_{\mathbf{k}}. \end{aligned} \quad (26)$$

Since we are looking for a linear renormalization group in  $v_s$ , it suffices to restrict ourselves to the Green's functions expanded to first order in  $v_s$ . Furthermore, the momentum-shell integral is for  $\mathbf{q}^2 = \Lambda^2 = 1$ , so that the Green's functions appearing in (26) are of the form

$$G[\mathbf{p} + \mathbf{k}]^{-1} = 1 + p - p_c + \sum_{s \geq 1} v_s \mathbf{k}^{2s} P_{(s)}^{(1)}(\dots, \theta_{\mathbf{k}}, \dots). \quad (27)$$

The usual percolation fixed point  $v_s = 0$  and  $p = p_c$  describes the geometrical properties of percolation clusters. These properties are completely independent of the electrical transport properties of the network. In other words, the field  $r_0$  is the coefficient of a polynomial of order 0 in  $\mathbf{k}$  so that the calculations of the fixed point and of the usual geometrical exponents  $\eta_p, \nu_p$  are independent of the values of the fields  $v_s$ , which are all associated with higher-order polynomials in  $\mathbf{k}$  times the field operator  $\Phi_{\mathbf{k}} \Phi_{-\mathbf{k}}$ . Hence one obtains the same results as in Ref. [10], namely,  $g^* = \epsilon/7$  for the fixed-point value of the coupling constant and  $\eta_p = -\epsilon/21$  and  $\nu_p = \frac{1}{2} + \frac{5\epsilon}{84}$  for the exponents. These exponents and fixed-point values of  $g$  and  $r_0$  come from what we call the first renormalization group, whose predictions relate to the geometrical properties of the percolation network and hence are independent of electrical properties.

Going further to obtain the exponents  $\psi_s$  associated with  $v_s$ , one has to linearize Eq. (25) in  $v_s$ . To the order we are working, the  $v_s$  are all multiplied by the same power  $\Phi_{\mathbf{k}}$ , namely,  $\Phi_{\mathbf{k}}^2$ , but by different powers of  $\mathbf{k}$ , namely, a homogeneous polynomial of order  $2s$ . Hence we consider the renormalization group as a renormalization equation in the space of homogeneous polynomials in  $\mathbf{k}$ . This is what we call the second renormalization group.

To be more specific, let us start with the simple case of  $d$  larger than the upper critical dimension ( $d_c = 6$ ). The renormalization group equations for  $r_{\mathbf{k}}$  then read

$$\frac{dr_{\mathbf{k}}}{dl} = 2r_{\mathbf{k}}. \quad (28)$$

In other words, the recursion relation for the  $v_s$  is obtained by linearizing around the fixed point  $v_s = 0$ ,  $s = 0, 1, \dots$ . To first order in  $v_s$  one has

$$r_{\mathbf{k}} = p - p_c + \sum_{s \geq 1} \mathbf{k}^{2s} v_s P_s(\dots, \theta_{\mathbf{k}_{\alpha, \beta}}, \dots) \quad (29)$$

and thus the scaling in Eq. (24) leads to

$$\frac{dv_s}{dl} = 2v_s. \quad (30)$$

Above six dimensions then, any  $P(\mathbf{k}) = \mathbf{k}^{2s} P_s(\dots, \theta_{\mathbf{k}_{\alpha, \beta}}, \dots)$  is an eigenvector of the second renormalization group. The eigenvalues are all identical, as is already known. [5] Thus, even though the precise form of  $P_s(\mathbf{k})$  depends on the microscopic noise distribution, the eigenvalue for  $v_s$  does not. Below six dimensions, things are less trivial since the self-energy  $\tilde{\Sigma}_{\mathbf{k}}$  depends functionally on  $P_s(\mathbf{k})$ . Nevertheless, we will show that there exists an eigenbasis of polynomials in  $\mathbf{k}$  whose eigenvalues are labeled only by the order of the polynomial. This means that whatever the starting  $P_s(\mathbf{k})$ , the eigenvalue for  $v_s$  depends only on  $s$ .

Below six dimensions, the renormalization group equations must be obtained by linearizing the self-energy  $\tilde{\Sigma}_{\mathbf{k}}$  as a function of  $v_s$ ,  $s \geq 2$ . We show in the Appendix that the eigenpolynomial basis is obtained, in the limit  $v_s \ll v_1$ , by solving for each integer  $s$  the equation

$$\begin{aligned} -2 \int_0^{+\infty} du \int_0^{+\infty} dt \exp\{-(u+t)(1+r_0)\} \left[ \frac{t}{t+u} \right]^{2s} u \\ \times \left\{ \exp\left( \frac{u+t}{4v_1} t^{-2} \Delta \right) P(\mathbf{k}) \right\} = d_s P(\mathbf{k}), \end{aligned} \quad (31)$$

which corresponds to the limits  $\sigma_0^{-1} \rightarrow 0$  and  $n \rightarrow 0$  for the self-energy. In Eq. (31) the Laplacian operator  $\Delta$  acts on  $\mathbf{k}$  and its limit, as  $n \rightarrow 0$ , is discussed in the next subsection. Because  $r_{\mathbf{k}}$  in Eq. (25) is linear in  $v_s$ , the eigenvalues for the second renormalization group are obtained as  $2 - \eta_p - g^*(2 + d_s)$ . In what follows,  $g$  will be taken at its fixed-point value  $g^*$ .

## B. Eigenvalues and eigenvectors of the second renormalization group

Equations (25) and (31) define the renormalization group equation for homogeneous polynomials of order  $2s$ . To diagonalize this equation we need eigenfunctions for the Laplacian operator. In  $n$  dimensions, the most general form for these eigenfunctions is given by [19]

$$k^{-\frac{n-2}{2}} J_{2r+\frac{n-2}{2}}(\gamma k) Y_{2r}(\dots, \theta_{\mathbf{k}}, \dots), \quad (32)$$

where  $Y_{2r}(\dots, \theta_i, \dots)$  is a spherical harmonic of order  $2r$  in  $n = NM$  dimensions and  $J_r$  is the Bessel function of order  $r$  with eigenvalue  $-\gamma^2$ . To expand polynomials of order  $2s$  on this basis, we first recall that the maximum value of the replica space  $k \equiv \sqrt{\mathbf{k}^2}$  is given by the ultra-violet cutoff  $\Lambda_k = k_{\max} = 2\pi/\Delta V_{\min} = \sigma_0 2\pi/i_{\min}$ . We can then proceed as follows. First note that an arbitrary polynomial of order  $2s$  can be written as

$$P_s(k_{\alpha\beta}) = \mathbf{k}^{2s} \sum_{0 \leq r \leq s} a_{s,r} Y_{2r}(\dots, \theta_{\mathbf{k}}, \dots), \quad (33)$$

where, to simplify the notation, we do not write internal indices related to the degeneracy of the spherical harmonics. Then, we write

$$\begin{aligned} \mathbf{k}^{2s} Y_{2r}(\dots, \theta_{\mathbf{k}}, \dots) &= \left( \mathbf{k}^{2s} - \mathbf{k}^{2r} \Lambda_k^{2(s-r)} \right) Y_{2r}(\dots, \theta_{\mathbf{k}}, \dots) \\ &+ \mathbf{k}^{2r} \Lambda_k^{2(s-r)} Y_{2r}(\dots, \theta_{\mathbf{k}}, \dots). \end{aligned} \quad (34)$$

The last term is an eigenfunction of the Laplacian with eigenvalue zero, while the first term can be expanded using the Fourier-Bessel expansion [20], which is uniformly valid for functions with zero value at the end of the interval

$$\begin{aligned} &\left( \mathbf{k}^{2s} - \mathbf{k}^{2r} \Lambda_k^{2(s-r)} \right) Y_{2r}(\dots, \theta_{\mathbf{k}}, \dots) \\ &= \sum_{l \geq 1} b_l k^{-\frac{n-2}{2}} J_{2r+\frac{n-2}{2}} \left( \frac{k}{\Lambda_k} \zeta_l \right) Y_{2r}(\dots, \theta_{\mathbf{k}}, \dots), \end{aligned} \quad (35)$$

where  $\{\zeta_\ell\}_{\ell \geq 1}$  are the zeros of the Bessel function of order  $2r + (n-2)/2$ .

We can now substitute this expansion in the kernel of the integral appearing in the recursion relation Eq. (31). The contribution of a general term of the series will have the form

$$\begin{aligned} &2 \int_0^\infty du \int_0^\infty dt \exp \left[ -\frac{u+t}{2v_1} t^{-2} \left( \frac{\zeta_l}{\Lambda_k} \right)^2 \right] \\ &\times k^{-\frac{n-2}{2}} J_{2r+\frac{n-2}{2}} \left( \frac{k}{\Lambda_k} \zeta_l \right) Y_{2r}(\dots, \theta_{\mathbf{k}}, \dots). \end{aligned} \quad (36)$$

The  $n = NM \rightarrow 0$  limit is obvious. Taking the  $\sigma_0^{-1} \rightarrow 0$  limit simplifies further the equation considerably since, in the argument of the exponential,  $[v_1 \Lambda_k^2]^{-1}$  is proportional to  $\sigma_0^{-1}$ . This means that effectively every term of the series behaves as if it had the same eigenvalue for the Laplacian, namely, 0. Since  $k/\Lambda_k$  is bounded between 0 and 1, the series can be resummed and any polynomial of order  $2s$  is an eigenvector with eigenvalue [10]

$$\begin{aligned} d_s &= -2 \int_0^{+\infty} du \int_0^{+\infty} dt \exp[-(u+t)] \left[ \frac{t}{t+u} \right]^{2s} u \\ &= -\frac{2}{(2s+1)(s+1)}. \end{aligned} \quad (37)$$

The main conclusion of the section is that the space of homogeneous functions of order  $2s$  can be characterized by a single eigenvalue that depends only on the order of the polynomial. The form of the microscopic distribution of the noise is therefore not relevant for scaling properties of the macroscopic resistance fluctuations. Only the condition  $v_s \ll v_1 \forall s > 1$ , which was used to derive Eq. (31), needs to be satisfied. The critical exponents associated with the fields  $v_s$  are thus

$$\psi_s = \nu_p [2 - \eta_p - g^*(2 + d_s)] = 1 + \frac{\varepsilon}{7(s+1)(2s+1)}, \quad s = 1, 2, \dots, \quad (38)$$

as found by Parks, Harris, and Lubensky ( $\nu_p = \frac{1}{2} + \frac{5}{84}\varepsilon$  and  $\eta_p = -\frac{\varepsilon}{21}$ ) [10]. It is shown in the Appendix that this formula applies also for the case  $s = 1$ , as written above.

## V. DISCUSSION

### A. Gap scaling

The scaling of the usual thermodynamic observables is normally trivially obtained from a few exponents only. This is usually referred to as gap scaling. Gap scaling also occurs in the present case. For example,

$$\left. \frac{\partial^\ell G_{\mathbf{k}}(x, x')}{\partial v_s^\ell} \right|_{v_s=0} \sim |x - x'|^{\ell \psi_s / \nu_p}. \quad (39)$$

That this applies to multifractals in percolation was verified by numerical simulations in [21]. Clearly, one can also define universal amplitude ratios [15,21].

### B. Symmetry breaking

With  $v_s = 0$  for all  $s$ , the action is invariant under the global transformation

$$\Phi'_{\mathbf{k}} = \Phi_{\mathcal{R}\mathbf{k}}, \quad (40)$$

where  $\mathcal{R}$  is a rotation (permutation) of the vector  $\mathbf{k}$  in the replica space of dimension  $MN$ . In other words, the action transforms according to the unit representation of the group  $O(MN)$ . When  $v_s \neq 0$ , that symmetry is broken since polynomials of higher degree transform like higher-dimensional representations of the group  $O(MN)$  [22]. To have every  $v_s$  associated with a different representation of the symmetry group is a necessary but not sufficient condition to have an infinity of observable exponents. Indeed, operators of different symmetry could

couple when higher-order corrections to the  $\epsilon$  expansion are evaluated. All this is analogous to what happens with symmetry-breaking fields in other critical phenomena models, such as the  $XY$  model for example [23]. The exponents  $\psi_s$  here are analogous to the crossover exponents  $\varphi_n$  of the  $XY$  model [24].

### C. Dominant exponents

At first sight, the perturbations associated with the  $v_s$  are all relevant since all the exponents  $\psi_s$  are found to be larger than zero. There are two important differences, however, with critical phenomena (say the case of the  $XY$  model).

(a) There is no physical realization that we know of for the lower-symmetry fixed point towards which the system rescales when one of the symmetry-breaking perturbations is different from zero. All physical observables are derivatives evaluated at a zero value of the symmetry-breaking fields  $v_s$ : In other words, the exponents  $\psi_s$  are crossover exponents associated with the symmetric fixed point.

(b) There is an additional freedom to rescale  $\mathbf{k}$  at each iteration as seen in Eq. (19). This allows one to formulate the renormalization group in such a way that only a finite number of operators are relevant. Indeed, for the usual percolation fixed point, the rescaling of the operators is found by choosing that the coefficient of the spatial gradient term in Eq. (18) to be a constant. Since the recursion relations for  $u_3$  and  $r_0$  are completely independent of  $\mathbf{k}$ , the geometric percolation fixed point is the usual one. The scale factor for  $\mathbf{k}$ , by contrast, may be chosen at will. This influences the recursion relations for the  $v_s$  and hence the corresponding  $\psi_s$  exponents. More specifically, the rescaling part of the renormalization group transformation may be written as

$$\Phi_{\mathbf{k}}(q) \longrightarrow \Phi'_{\mathbf{k}' \equiv a\mathbf{k}}(q' \equiv qb) = b^{(-d-2+\eta_p)/2} \Phi_{\mathbf{k}}(q). \quad (41)$$

As an example, we choose the scale factor  $a$  such that the total resistance is kept constant under rescaling of all lengths by a factor  $b$ . This is done by first noting that after eliminating the degrees of freedom, the scaling of  $v_s$  is obtained by keeping the corresponding terms of the action invariant

$$\begin{aligned} v'_s b^{g^*(2+d_s)} k'^{2s} [\Phi'_{\mathbf{k}'}(q')]^2 (\Delta k')^{MN} d^d q' \\ = v_s k^{2s} [\Phi_{\mathbf{k}}(q)]^2 (\Delta k)^{MN} d^d q. \end{aligned} \quad (42)$$

In the limit  $NM \rightarrow 0$ ,  $(\Delta k)^{NM}$  does not come in since the infinite-system limit ( $\Delta k = 0$ ) is taken last. Setting  $a = b^y$  and  $b = e^l$ , we obtain

$$\frac{dv_s}{dl} = [2 - \eta_p - g^*(2 + d_s) + 2sy] v_s. \quad (43)$$

Choosing

$$y = -\frac{1}{2} [2 - \eta_p - g^*(2 + d_1)] \quad (44)$$

to keep the voltage across the network constant (i.e.,  $v_1$  does not scale), the exponents associated with  $v_s$  become

$$\psi_s \rightarrow \psi_s - 2s\psi_1 \quad (45)$$

so that the fields  $v_s$  now appear irrelevant for  $s \geq 2$ . In other words, since the  $\psi_s$  are a decreasing function of  $s$ , we may always choose the scaling dimension of  $\mathbf{k}$  such that only a few of the  $\psi_s$  exponents are positive, without influencing the physics. These statements can be rephrased in a more physical way by recognizing that the field theory of PHL corresponds to computing the power dissipated between points  $x$  and  $x'$  when a unit current is injected between these points, whatever the distance between  $x$  and  $x'$ : One could just as well decide to rescale at unit voltage instead of unit current and this would correspond to multiplying  $\mathbf{k}$  by a scale factor at each iteration. The size dependence of  $\Lambda_k$  does not matter since, in the limit  $\sigma_0^{-1} \rightarrow 0$ ,  $\Lambda_k$  goes to infinity independently of the system size. The rescaling in  $\mathbf{k}$  is associated with the scaling of applied voltage so that the scalings in  $\mathbf{k}$  and  $\mathbf{q}$  space are independent of each other.

The remarks of the preceding paragraph may also be formulated as follows: The analysis that we have done to find the eigenpolynomials for Eq. (25) shows that the latter equation is like a second renormalization group with fields  $v_s$ , which describe the electrical properties of an object whose (critical) geometrical properties are given by the first renormalization group, with fields  $r_0$  and  $u_3$ . The first renormalization group has properties totally independent of those of the second, while the second is slaved to the first. The  $v_s$  are conjugate to replica-space gradients of the field operators  $\Phi$ . In other words, they are conjugate to polynomials of degree  $2s$  in  $\mathbf{k}$  times  $\Phi_{\mathbf{k}}^2$ . The rescaling of  $\mathbf{k}$  in the second renormalization group is arbitrary and this is fundamentally due to the linearity of Kirchhoff's laws. Instead of referring to relevant or irrelevant exponents, for that second renormalization group it makes more sense to call them dominant exponents since one expects that the observables that are connected to  $\psi_s$  are also coupled to other operators giving corrections to scaling (subdominant exponents).

### D. Observability

The  $\varphi_n$  of the  $XY$  model are not all relevant exponents. In fact, for  $n \geq 4$ , they correspond to irrelevant operators [1,23–25]. While they are only a subset of all possible irrelevant exponents, they are, however, special because, for increasing values of  $n$ , they represent the leading scaling behavior of operators with lower and lower symmetry. It is their symmetry instead of their relevance that seems fundamental for their observability. The same remark applies for the dominant exponents  $\psi_s/\nu \equiv -x_n$  discussed above.

## VI. CONCLUSION

We have shown that the field theory for multifractals in percolation has a special structure that allows multi-

fractal exponents to have properties that do not usually appear in standard critical phenomena. They follow from symmetry-breaking operators in a second renormalization group to which an additional normalization freedom (e.g., scaling at constant voltage or constant current) is associated. This freedom allows one to arbitrarily shift the crossover exponents (while maintaining the observable quantities unchanged). We propose to call these exponents dominant since, even though their value can be shifted, they are trivially related to the leading scaling behavior of operators characterized by a given symmetry. A second renormalization group also occurs in dynamical systems [12] and probably also in the field of localization.

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**APPENDIX: RECURSION RELATION TO ONE LOOP**

In this appendix we derive the renormalization group equation (31). Let us recall that, to one-loop order, the recursion relation for  $r_{\mathbf{k}}$  takes the form

$$\frac{dr_{\mathbf{k}}}{dl} = (2 - \eta_p) r_{\mathbf{k}} - g(2 + \tilde{\Sigma}_{\mathbf{k}}), \tag{A1}$$

where the self-energy is defined by

$$\tilde{\Sigma}_{\mathbf{k}} = \lim_{n \rightarrow 0} \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} dp_{\alpha\beta} [G(\mathbf{p} + \mathbf{k}) G(\mathbf{p})] \tag{A2}$$

with

$$[G(\mathbf{k})]^{-1} = 1 + p - p_c + \sum_{s \geq 1} v_s k^{2s} P_s^{(1)}(\dots, \theta_{\mathbf{k}}, \dots). \tag{A3}$$

Equation (A1) is a nonlinear recursion relation for the  $v_s$ . We now show how one can linearize this recursion relation in the limit where  $n \rightarrow 0$  and  $\sigma_0^{-1} \rightarrow 0$ . Using the Schwinger representation for the propagator

$$G(\mathbf{k}) = \int_0^{+\infty} du \exp\{-u[G(\mathbf{k})]^{-1}\} \tag{A4}$$

we get for Eq. (A2)

$$\lim_{n \rightarrow 0} \int_0^{+\infty} du \int_0^{+\infty} dt \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} dx_{\alpha\beta} \times \exp\left\{-u[G(\mathbf{x})]^{-1} - t[G(\mathbf{x} + \mathbf{k})]^{-1}\right\}. \tag{A5}$$

By making the change of variables

$$\mathbf{x} \rightarrow \mathbf{x} - \left[\frac{t}{t+u}\right] \mathbf{k} \tag{A6}$$

Eq. (A5) can be expressed as

$$\lim_{n \rightarrow 0} \int_0^{+\infty} du \int_0^{+\infty} dt \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} dx_{\alpha\beta} \exp\left\{-uG\left[\mathbf{x} - \left(\frac{t}{t+u}\right) \mathbf{k}\right]^{-1} - tG\left[\mathbf{x} + \left(\frac{u}{t+u}\right) \mathbf{k}\right]^{-1}\right\}. \tag{A7}$$

For  $v_s \ll v_1$  we can expand to first order in  $v_s$  all terms in  $G$  that are not invariant under rotation (the only term that is invariant is  $\mathbf{k}^2$  and it is associated with the resistance). It is useful to define

$$P \equiv \sum_{s \geq 2} v_s \mathbf{k}^{2s} P_s(\dots, \theta_{\mathbf{k}}, \dots). \tag{A8}$$

Therefore, Eq. (A7) reads

$$-\lim_{n \rightarrow 0} \int_0^{+\infty} du \int_0^{+\infty} dt \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} dx_{\alpha\beta} \exp\left[-(u+t)(1+p-p_c) - v_1 \mathbf{x}^2 (u+t) - v_1 \frac{tu}{t+u} \mathbf{k}^2\right] \times u \left\{ P\left[\mathbf{x} - \left(\frac{t}{t+u}\right) \mathbf{k}\right] + P\left[\mathbf{x} + \left(\frac{t}{t+u}\right) \mathbf{k}\right] \right\}. \tag{A9}$$

To interpret Eq. (A9) observe that in dimension  $n$

$$\int_{-\infty}^{+\infty} dx_1 \dots \int_{-\infty}^{+\infty} dx_n \exp\left[-v_1(u+t)(x_1^2 + \dots + x_n^2)\right] x_{\beta_1}^{2p_1} \dots x_{\beta_j}^{2p_j} = \left(\frac{\pi}{v_1(u+t)}\right)^{\frac{n}{2}} \prod_{i=1}^{i=j} \left\{ \frac{(2p_i)!}{2^{p_i} p_i!} \left[\frac{1}{2v_1(u+t)}\right]^{p_i} \right\}, \tag{A10}$$



which gives as  $n \rightarrow 0$

$$\prod_{i=1}^{i=j} \left\{ \frac{(2p_i)!}{2^{p_i} p_i!} \left[ \frac{1}{2v_1(u+t)} \right]^{p_i} \right\}. \quad (\text{A11})$$

For analytic functions  $P$ , we can write with  $\mathbf{k}' = \frac{t}{u+t} \mathbf{k}$

$$P(\mathbf{k}' - \mathbf{x}) + P(\mathbf{x} + \mathbf{k}') = 2 \sum_{l \geq 0} \sum_{p_1 + \dots + p_j = l} \frac{1}{(2p_1)! \dots (2p_j)!} x_1^{2p_1} \dots x_j^{2p_j} \frac{\partial^{2p_1 + \dots + 2p_j} P}{\partial k_1^{2p_1} \dots \partial k_j^{2p_j}}. \quad (\text{A12})$$

The integrals can then be evaluated as

$$\begin{aligned} & \lim_{n \rightarrow 0} \int_{-\infty}^{+\infty} d^n \mathbf{x} \exp[-v_1(u+t)\mathbf{x}^2] [P(\mathbf{x} - \mathbf{k}') + P(\mathbf{x} + \mathbf{k}')] \\ &= 2 \sum_{l \geq 0} \frac{1}{[4v_1(u+t)]^l} \sum_{p_1 + \dots + p_j = l} \frac{1}{p_1! \dots p_j!} \frac{\partial^{2p_1 + \dots + 2p_j} P}{\partial k_1^{2p_1} \dots \partial k_j^{2p_j}} = 2 \exp \left[ \frac{\Delta_{\mathbf{k}'}}{4v_1(u+t)} \right] P(\mathbf{k}') = 2 \exp \left[ \frac{u+t}{4v_1} t^{-2} \Delta \right] P(\mathbf{k}), \end{aligned} \quad (\text{A13})$$

where  $\Delta$  is the Laplacian operator for the variable  $\mathbf{k}$ . In this equation we have used the fact that for  $s \geq 2$  the term  $v_1 \frac{tu}{t+u} k^2$  in (A9) cannot play any role to linear order in the  $v_s$ . Now that we have interpreted Eq. (A9), we can substitute it into Eqs. (A7), (A5), and (A2) to obtain

$$\begin{aligned} \lim_{\sigma_0^{-1} \rightarrow 0} \lim_{n \rightarrow 0} \tilde{\Sigma}_{\mathbf{k}} &= - \lim_{\sigma_0^{-1} \rightarrow 0} \sum_{s \geq 2} 2v_s \int_0^\infty du \int_0^\infty dt \exp \left[ -(u+t)(1+r_0) \right] \left[ \frac{t}{u+t} \right]^{2s} u \\ &\times \left\{ \exp \left[ \frac{u+t}{4v_1} t^{-2} \Delta \right] P_s(\mathbf{k}) \right\}. \end{aligned} \quad (\text{A14})$$

Because the thermodynamic limit is taken last in replica approaches, we can show, as is done in the text, that  $\exp \left[ \frac{u+t}{4v_1} t^{-2} \Delta \right]$  has a well-defined limit as  $\sigma_0^{-1} \rightarrow 0$ .

The case  $s = 1$  can now be treated simply. Indeed, in this case, it is only the term  $v_1 \frac{tu}{t+u} k^2$  that plays a role. Expanding it, the term linear in  $v_1$  in  $\tilde{\Sigma}_{\mathbf{k}}$  is given by

$$\lim_{\sigma_0^{-1} \rightarrow 0} v_1 \int_0^\infty du \int_0^\infty dt \exp[-(u+t)] \frac{tu}{t+u} \mathbf{k}^2.$$

This shows that  $\mathbf{k}^2$  is an eigenpolynomial, as quoted in the text. Furthermore, the corresponding eigenvalue  $d_1$  does correspond to the  $s = 1$  limit of Eq. (38).

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